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Vertex covering under constraints

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Introduction

The life cycle of a PhD thesis contains a step so frightening that students lose sleep. I am of course talking about the choice of the meals for the post-defense party. Dealing with people that don't like spicy food, those who are lactose intolerant, those who only like sweets... these parties are real brain-teasers. Fortunately, I already addressed this problem by making it the subject of my PhD thesis. No need to look back at the front page, the title of my PhD is not "About the issue of PhD parties : comparative analysis of academics' tastes" (this title was duly refused by the doctoral school). As I wasn't able to choose this more adequate title I had to get around this problem and name my PhD "Vertex covering under constraints". However, the reader should not be fooled, these are the same subjects and here is the description.

On the one hand, there is the set of guests to my PhD party (not counting students attracted by the smell of free food [1]). On the other hand, there is the set of possibles meals : spicy food, sweets, dairy products... Moreover, there is a relation between guests and meals allowing us to know what meals are liked by each guest. The goal is that each guest finds at least one thing they like at the party. An obvious solution would be to bring every existing meal to the party. Sadly, for financial reasons - and because I doubt the university will lend me a room big enough - this solution is not possible. In order to please my bank account, we can reformulate this problem by asking what is the smallest number of meals that needs to be brought to the party so that everyone's palate is satisfied. Figure 1 depicts a possible situation.

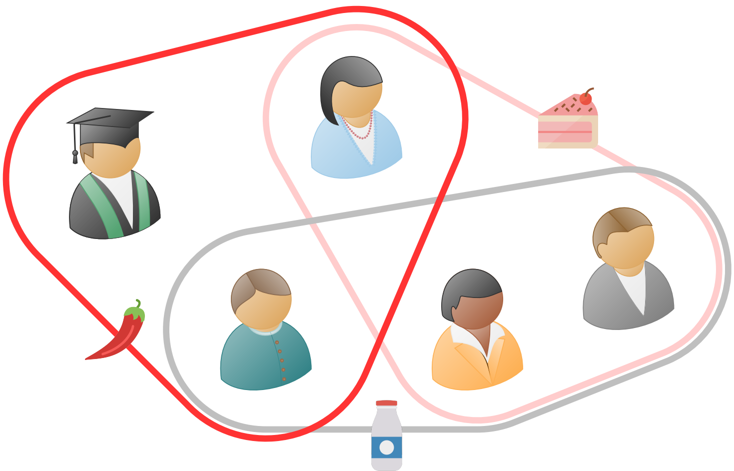

FIGURE 1 – Distribution of the guests' tastes during a PhD party. Spicy foods and sweets are enough to please every guests.

In combinatorics, this problem is called *set cover*. It is a classical computer science problem studied since 1972. The instances of this problem are composed of finite sets of elements (our guests) and of sets of attributes, subsets of elements (for example which guests like to eat spicy food). The goal is to select a minimal number of attributes such that each element is part of at least one of the selected attributes. If this problem has an application in the resolution of the PhD party problem, it is also used in other fields with, in particular, many applications in network monitoring. If the elements of an instance of the set cover problem correspond to the nodes of a network and if we want to put sensors to make sure there is no failure, then the attributes can represent the nodes that each possible sensor can monitor. Choosing the minimal number of attributes to hit all the vertices is equivalent to put the minimal number of sensors to make sure that all the nodes are monitored.

Because of the many forms attributes can take, the set cover problem is very difficult to solve in the general case. However, the attributes given to the vertices are often characterized by an underlying structure. If we come back to the PhD thesis party problem, it is likely that individuals of the same family like the same meals or that kids like sweets. In these cases, the attributes would be linked to family relations and to classes of ages. Likewise, for the case of network monitoring, a sensor is not able to monitor two vertices at different extremities of a network. *Graphs* are one of the structure that this problem can take. Broadly, a graph is a set of points, named vertices, connected by lines, named edges.

1. Did you know ? It has been proven that a PhD student is able to smell free food in a 500m radius, even more around noon.

The structure of a graph is a lot more constrained than general instances of the set cover problem and, when the attributes of the set cover problem follow this structure, we can sometimes achieve better results than in the general case.

Let's take, for example, the *edge cover problem*. In this problem, we are looking for the minimal number of edges of a graph such that each vertex is the extremity of at least one of the selected edges. This problem is equivalent to the set cover problem for the case where each attribute is associated to exactly two elements. This problem has been solved since the end of the seventies. Defined in a similar way, the *clique cover problem*, in which we try to cover the vertices of a graph by using complete subgraphs, is equivalent to the set cover problem in the case where all the cliques of a graph become the attributes. The edge cover problem is actually a subcase of the clique cover problem in which all the cliques are of order 2.

Another well studied covering problem is the *domination problem* in which we try to find a *dominating set* of minimal size. A dominating set is a subset of vertices of a graph such that every vertex is either a part of this set or has a neighbor in this set. The domination problem corresponds to the instances of set cover problem in which the attributes are the closed neighborhoods of a graph. The *total domination problem* is similar, the only difference being that the selected vertices must also have a neighbor in the set. Therefore, this problem corresponds to the instances of set cover where the attributes are open neighborhoods.

Another way to obtain an instance of set cover from a graph is the *isometric path number problem* in which we are looking for a way to cover the vertices of a graph by using shortest paths in the graph. The attributes are the sets of vertices that are on shortest paths of the graph.

These problems have been studied for many years and there is an extensive literature about them. They also lead to variations which are beyond the scope of set cover. Going back to the case of the PhD party, we can, for example, imagine a situation in which some guests are more popular than others and that the meals they eat will also be liked by other guests because of a "group effect", leading to a propagation of the taste for some meals. On the contrary, some PhD students in opposition crisis with their advisers might not want to eat the same thing as them. Then, we need to cover them with different sets of meals. These are only a few of the problems that PhD students preparing their PhD party have to face. Similarly, every problem described above has many variations to be adapted to their different applications. In this manuscript, we will precisely study various problems which are variations of the set cover problem.

In the first chapter, we introduce graph and complexity notions which will be used afterwards. Using these notions, we define more formally covering problems and in particular the domination problem. Indeed, chapters two and three concern variants of the domination problem. The second chapter of this PhD concerns the *power domination problem*. The power domination is a variant of the domination problem in which a propagation process is added. In this problem, the goal is to *monitor* all the vertices of a graph. Initially some vertices are selected and their neighborhood is monitored. This monitoring is then propagated the following way : if a vertex is monitored and if all of its neighbors except one are also monitored, then the monitoring is propagated to this remaining vertex. The propagation continue until either every vertex is monitored or there are no new vertex that can be monitored using the previous rule. A *power dominating set* is a set of vertices such that if this set is initially selected then all the vertices of the graph end up monitored. We study this problem on grids, graphs with a high regularity. We give the minimal size of a power dominating set for triangular grids with triangular shape and also give an upper bound for square grids of dimension 3.

In the third chapter, the change is not in the covering method but in the type of problem. Indeed in this chapter we study the Maker-Breaker domination game, a two-player game on a graph. In this game, two players, Dominator and Staller, alternatively select vertices of a graph. Dominator wants the set of vertices that he selected to dominate the graph while Staller wants to stop him from doing so. This game can model a situation in which we want to monitor a network but some failures happen, keeping us from placing sensors on some nodes. We study this game using two complementary approaches : on one hand we study the winner of the game (which player has a winning strategy) and on the other hand we study the number of moves that Dominator needs to play in order to win if he has a winning strategy. In this perspective, we introduce another graph covering method ; the pairing domination. The existence of a pairing dominating set ensures that Dominator has a winning strategy and gives an upper bound on the number of moves that he needs to play. However, we prove that it is difficult to know if a pairing dominating set exists and that the absence of such a set is not enough to ensure that Dominator has no winning strategies. We then prove that determining the winner of the game is hard in the general case but that it is easy for trees and cographs. Finally, we generalize this problem to the case of the total domination and to an Avoider-Enforcer version.

In the fourth chapter, we move away from the domination problem and grow closer to the isometric path number problem, *i.e.* the covering by shortest paths. Indeed, in this chapter we study the *strong* *geodetic number*. In this problem, the goal is to cover a graph by using shortest paths but instead of directly choosing these paths, we choose their extremities. More precisely, *strong geodetic set* is a set of vertices of a graph such that there exists a way to fix a shortest path between each pair of selected vertices in such a way that each vertex of the graph is on at least one of these paths. The *strong geodetic number* is the size of a minimal strong geodetic set. We mainly study this problem in relation to the cartesian product. To this end, we introduce the *strong geodetic core* which allows us to improve existing bounds. We also apply this notion to study hypercubes.

As previously mentioned, these first three chapters are close to the set cover problem in the case where the structure of the instances of the problem is a graph. However, graphs are not the only structure worthy of interest. For example, if the elements are points in a metric space, it is more relevant to have a geometrical approach. It is then natural to cover vertices with a set of disks. Indeed, if a sensor is put on a point of the plane it seems reasonable to assume this sensor has a radius of action and that it covers the surface of a disk. Thus, in the fifth and last chapter, we cover elements of the plane with a set of disks. Another constraint is also added, we want that this set of disks to *identify* the vertices. Coming back to the sensors with a radius of action, if a sensor spots an anomaly, it does not necessary know from which direction this anomaly comes from. However, if a certain number of sensors detect an anomaly at the same time, it must be at the intersection of their radiuses of action. This is the principle of the *identification of points in the plane with disks*. In the presence of a certain set of points in the plane, we want to find a set of disks such that each point is inside at least a disk and that, for each point, the set of disks containing it is unique. We study this problem for specific configurations and give bounds in the general case. We also study the complexity of this problem when the radius of the disks is fixed.

We can also note that some of my works do not fit in the scope of this PhD thesis and therefore are not cited in this manuscript. Indeed, some of my past and present works deal with combinatorial games and can not be interpreted in a covering context. Among those, one can find the article "A generalization of Arc-Kayles" [2] that I wrote in collaboration with my PhD fellows, Antoine Dailly et Marc Heinrich.

This abstract of my thesis sums up the main results present in my manuscript and the links between them.

The links to the articles can be found on my web page : `https://perso.liris.cnrs.fr/valentin.gledel/research.html`

2 Power dominating set

The main result of this chapter is the power domination number of triangular grids of triangular shape. It comes from the article "Power domination on triangular grids with triangular and hexagonal shape" [1] written in collaboration with Bose, Pennarun and Verdonschot.

The other result concerns the power domination number of square grids of dimension 3 and comes from my personal work.

In this chapter, we study power domination; a variant of the domination problem in which there is a propagation of the domination, thus allowing to decrease the number of vertices that have to be selected.

Definition 2.1. *Let G be a graph and $S \subseteq V(G)$ be a subset of vertices of G. The set M of* monitored *vertices is defined as follows. At first M is formed by the closed neighborhood of S, then we iteratively add vertices in M if they respect the following rule : a vertex u is added to M if it has a neighbor v such that $N[v] \setminus u \subseteq M$. The set S is a* power dominating set *of G is at the end of the propagation process $M = V(G)$. We call* power dominating number *of G the size of a smallest power dominating of G. This parameter is noted $\gamma_P(G)$.*

Power domination has been studied on different types of grids and products of paths. The contributions of this chapter are in the context of this thematic. We first present a result on the power dominating number of triangular grids with triangular shape :

Theorem 2.2 (Bose, G., Pennarun et Verdonschot, 2018+ [1]). *If T_k is a triangular grids with triangular shape of size $k-1$, with $k \in \mathbb{N}^*$, then $\gamma_P(T_k) = \left\lceil \frac{k}{4} \right\rceil$.*

The upper bound is obtained by exhibiting power dominating sets reaching it, as illustrated in Figure 2. The lower bound is shown using a quantity that can only decrease with the propagation and for which we study the value before and after the propagation.

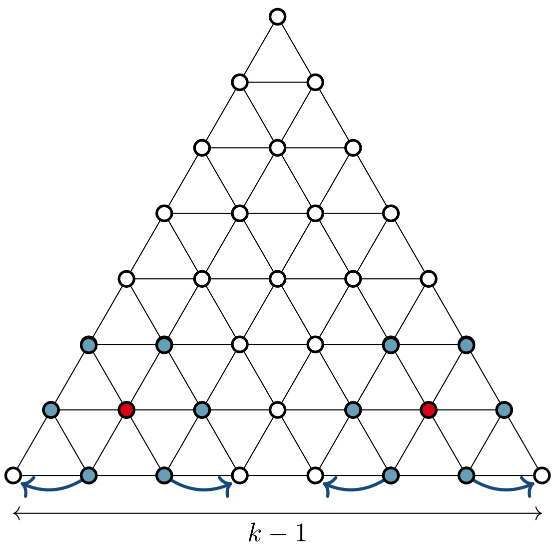

Figure 2 – A minimal power dominating set for the grid T_8. Vertices in red are in S and vertices in blue are in $N(S)$. After the propagation of four vertices, symbolized by blue arrows, the line $x = 0$ is fully monitored, allowing the propagation to cover the rest of the grid.

This chapter is concluded by the study of square grids of dimension 3, a work that is not yet the object of a publication :

Theorem 2.3 (G., 2019+). *There exists $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$, $\gamma_P(G_{n,n,n}) < \frac{n^2}{5}$.*

This result on square grids of dimension 3 is not an exact value, contrary to the result on triangular grids. A first step in the improvement of this result would be to find a satisfying lower bound.

3 Maker-Breaker domination games

The results of this chapter comes from three distinct papers :

The article "Maker-Breaker domination game" [3] introduces the Maker-Breaker domination game, the pairing dominating sets and gives complexity results.

The article "Maker-Breaker domination number" [8] focuses on the number of moves needed for Dominator to win the game. We give some results on the general case, show that the determining the Maker-Breaker domination number is PSPACE-complete even in games for which Dominator has a winning strategy (this is actually not in the article and was proved later). We also showed that it could be determined on trees in polynomial time.

The article "Maker-Breaker total domination game" [5] concerns a variation of the Maker-Breaker domination game in which Dominator try to build a total dominating set. The article give several results but in the manuscript I only focused on the proof this game is polynomial on cacti.

I also worked on my own on an Avoider-Enforcer version of the game and showed that this game is polynomial on trees, using a method similar to the Maker-Breaker version.

When we want to optimize a parameter of a graph, we imagine that we are in a perfect situation with no outside interventions. However, in most real life situations, random events can make the task more difficult. One way to model this is to consider our problem as a game in which we face an opponent with goals opposite to ours. Therefore, finding a strategy to beat this opponent in every possible game allows us to protect ourselves against the worst possible situations.

In this chapter, we will apply this principle to the domination problem by introducing the Maker-Breaker domination game.

The Maker-Breaker domination game played on a graph G is a Maker-Breaker game played on a hypergraph $\mathcal{H}$ with the same vertices as G and whose hyperedges are the closed neighborhoods of the graph. In other words, two players, Dominator and Staller, alternatively select vertices of G. Dominator wins if he succeeds in forming a dominating set (he has the role of Breaker) and Staller wins if she selects the vertices of a closed neighborhood (she has the role of Maker). In this chapter, we rarely use the general Maker-Breaker aspect of the game and mostly focus on the notion of dominating sets in graphs.

We have two main goals in the study of this game. The first one is, given a graph G, to find the outcome of the Maker-Breaker domination on G. This was the object of the seminar article on the subject [3], written in collaboration with Eric Duchêne, Aline Parreau and Gabriel Renault. The other goal that we have is to find the minimal number of moves needed for Dominator to win (for his winning positions). If Dominator starts the game, we call this number the D-Maker-Breaker domination number and if Staller starts we call it the S-Maker-Breaker domination number. These parameters were introduced in an article written in collaboration with Vesna Iršič and Sandi Klavžar in 2019 [8].

Pairing dominating set

The *pairing strategy* is a well-known winning strategy for Breaker in Maker-Breaker. This strategy can be applied when a subset of vertices can be partitioned into pairs such that each hyperedge contains at least one of the pairs. In this case, Breaker can win by always playing in the same pair as Maker and thus playing at least once in every hyperedge. In the context of the Maker-Breaker domination game, such a subset corresponds to a variation of the domination that we introduced and that is defined below :

Definition 3.1. *Let $G = (V, E)$ be a graph, A set of pairs of vertices $\{(u_1, v_1), \dots, (u_k, v_k)\}$ of V is a* pairing dominating set *if the pairs are pairwise disjoint and if the intersection of the closed neighborhoods of each pair cover all the vertices of G :*

$$V = \bigcup_{i=1}^{k} N[u_i] \cap N[v_i].$$

If a graph G has a pairing dominating set, then Dominator always has a winning strategy by playing first or second. However we show that it is not easy to know if a graph contains such a set :

Theorem 3.2 (Duchêne, G., Parreau et Renault, 2018+ [3])**.** *Deciding if a graph G contains a pairing dominating set is* NP*-complete.*

Moreover, there are some graphs for which Dominator has a winning by playing second or first and that do not have a pairing dominating set, see Figure 3 for examples of such graphs.

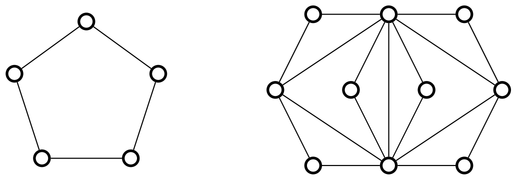

FIGURE 3 – Two graphs for which Dominator as a winning strategy and that do not have pairing dominating sets.

Complexity

Knowing that it is already hard to decide if a graph contains a pairing dominating sets, one might think that deciding the outcome the Maker-Breaker game is at least as difficult. Indeed, we have shown, using a reduction to Maker-Breaker, that the Maker-Breaker domination game is PSPACE-complete.

Theorem 3.3 (Duchêne, G., Parreau et Renault, 2018+ [3])**.** *Deciding the outcome of the Maker-Breaker domination game is* PSPACE*-complete on bipartite graphs.*

A similar proof can be applied to show the game is also PSPACE-complete on split graphs.

Concerning the Maker-Breaker domination numbers, one could argue that, since deciding outcome of game is PSPACE-complete, it is also PSPACE-complete to decide if the Maker-Breaker domination numbers are finite or not, let alone less than a certain value. However it feels like getting round the problem and not solving it. We show then that determining the Maker-Breaker domination number is PSPACE-complete even if we know that Dominator has a winning strategy :

Theorem 3.4 (Gledel)**.** *Deciding if the Maker-Breaker domination number is less than an integer k is* PSPACE*-complete on graphs that admits a pairing dominating set.*

However, we showed that finding the outcome of the Maker-Breaker domination game is polynomial for cographs and trees. To do so, we studied the outcome of the union of cographs, the join of two graphs and on another operation that we called "glue". The study of the glue operation shows that we can remove pendant P_2's from graphs without changing the outcome which gives the result on trees.

With similar methods we showed that it is also polynomial to determine the Maker-Breaker domination number of trees. However it is not the case for cographs as the Maker-Breaker domination number of the union and the join is yet to be found.

Other variations

A total domination variation has been defined in collaboration with Michael Henning, Vesna Iršič and Sandi Klavžar, leading to the publication of an article [5]. Several results were presented in this paper but in the scope of this PhD we only show that finding the outcome of this game is polynomial on cacti. The proof of this result is close to the proof of polynomiality for trees in the Maker-Breaker domination game with a few additional subtleties.

Moreover, I also worked on the Avoider-Enforcer of the game and showed that once again the game is polynomial for trees.

4 Strong geodetic number

This chapter covers the results from two different articles. The first one, "Strong geodetic cores and Cartesian product graphs" [7], deals with the strong geodetic number of the Cartesian product of two graphs. We improve the previous upper bound and disprove a previous conjecture on the lower bound. The other article, "Strong geodetic number of complete bipartite graphs, crown graphs and hypercubes" [6], deals with the strong geodetic number of three classes of graphs. This manuscript focused on two of them, complete bipartite graphs and hypercubes. These two articles are the results of work with Vesna Iršič and Sandi Klavžar during a research stay in Slovenia during Spring 2018.

The previous two chapters were about variations on the domination problem. It is not, however, the only type of vertex covering in graphs. In this chapter, we are interested in the covering of vertices using shortest paths.

Several parameters are linked to the covering by shortest paths (also called *geodesics*), in this chapter we focus mainly on the *strong geodetic number* comparing it to a very close parameter called the *geodetic number*. The geodetic number was introduced more than twenty years ago [10] and has been the subject of numerous studies while the strong geodetic number was only introduced recently [12]. The definition of this parameter is the following :

Definition 4.1 (Manuel *et al.* , 2018 [12])**.** *Let $G = (V, E)$ be a graph, $S \subseteq V$ is a* strong geodetic set *of G if there is a function $\widetilde{I} : \binom{S}{2} \to \mathcal{P}(V)$ that associates to each pair of vertices (u, v) of S a geodesic between u and v in such a way that $\bigcup_{u,v \in \binom{S}{2}} \widetilde{I}(u, v) = V$.*

The size of the smallest strong geodetic set of a graph is called the strong geodetic number *of the graph, denoted by* $\mathrm{sg}(G)$*.*

The first result presented in this chapter concerns the complete bipartite graph. We give a complete characterization of the strong geodetic number.

The rest of this chapter focuses on the relation between the strong geodetic number and the Cartesian product. In a previous paper on the topic, Iršič et Klavžar presented the following conjecture about the lower bound of the Cartesian product of two graphs (conjecture that they verified computationally for graphs with a small number of vertices) :

Conjecture 4.2 (Iršič et Klavžar, 2018 [11])**.** *If G is a graph with $|V(G)| \geq 2$, then* $\mathrm{sg}(G \,\square\, K_2) \geq \mathrm{sg}(G)$.

This conjecture seemed natural because its equivalent for the geodetic number is true and easy to prove, and also because intuitively the Cartesian product of a graph and K_2 "complicates" the the graph and should therefore induce a greater strong geodetic number. However, we proved that $\mathrm{sg}(G)$ can actually be bigger than $\mathrm{sg}(G \,\square\, K_2)$ and that the difference between the two can be arbitrary large. Figure 4 gives the example of a graph such that $\mathrm{sg}(G) > \mathrm{sg}(G \,\square\, K_2)$. In this section, we also study classes of graphs for which the conjecture holds true.

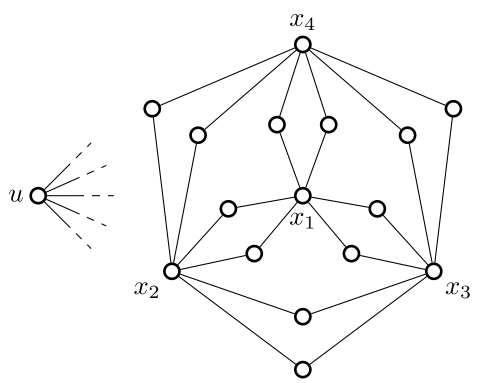

FIGURE 4 – A counterexample to the conjecture 4.2

Next, we introduce a new parameter : the *strong geodetic core*. The parameter can be interpreted as of how efficient a strong geodetic set is. If the strong geodetic core is high, this means that most of the vertices of the minimal strong geodetic set are "useful" and if it is low only a few are really needed do most of the covering. This parameter allows us to improve previous upper bounds on the strong geodetic number of the Cartesian product of graphs and, with similar ideas, to give an upper bound on the strong geodetic number of hypercubes.

5 Identification of points using disks

In this chapter, we cover the results of the article "Identification of points using disks" [9] written with Aline Parreau. This article (and therefore its associated chapter) deals with a problem similar to identifying codes or to test cover but with a geometrical constraint : the points are located in the plane and we want to identify them with disks.

In this last chapter, we are interested in a covering problem in which we want - in addition to the covering - to identify elements. This is relevant in monitoring problems for which we don't only want to know if there is failure or not (what covering is doing) but also where the failure occurred.

To answer this problematic in graphs, one way to do this is to consider *identifying codes*. An identifying code is a set C of vertices of a graph such that each vertex of the graph has a unique non-empty neighborhood in C. This definition can be generalized for hypergraphs.

With Aline Parreau, we wanted to add a geometric component to this problem. It seems natural to imagine that sensors can detect failures around them up to a certain distance, as in the case of smoke detectors for example. We can model this problem by points in the plane which represent areas likely to have a failure and by disks whose centers are the possible locations of the sensors. Here is a formal definition of this problem :

Definition 5.1. *Let $\mathcal{P}$ be a set of points of $\mathbb{R}^2$. A disk of radius $r \in \mathbb{R}$ and of center $c \in \mathbb{R}^2$ is the set of points of $\mathbb{R}^2$ at distance at most r from c. A point $P \in \mathcal{P}$ is* covered *by a disk if it belongs to this disk. Two points P and Q are* separated *by a disk D if exactly one of them is covered by D. A set of disks $\mathcal{D}$* identify *$\mathcal{P}$ if each point is covered and each pair of points is separated by $\mathcal{D}$.*

The smallest size of set of disks identifying a set $\mathcal{P}$ of points is denoted by $\gamma_D^{ID}(\mathcal{P})$.

Figure 5 gives an example of an identifying set of disks.

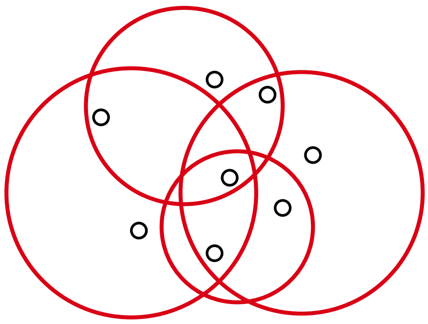

FIGURE 5 – A set of four disks identifying eight points.

The first section of this chapter considers points in particular configurations. We give an exact result for colinear points and an upper bound for grids.

We then study bounds on this problem. Results from other studies allowed us to find an upper and a lower bound form this problem, both tight. It is to be noted that the upper bound is almost exactly a result from Gerbener and Tóth [4] on the study of the separation of points by convex shapes. One of the main results of this chapter is that we improve the upper bound (which is of order $\frac{n}{2}$) when points are in general configuration - i.e. when no three points are colinear and no four points are cocyclic :

Theorem 5.2 (G. et Parreau, 2019 [9])**.** *If $\mathcal{P} \subseteq \mathbb{R}^2$ is a set of n points in general configuration, then $\gamma_D^{ID}(\mathcal{P}) \leq 2\lceil \frac{n}{6} \rceil + 1$.*

The proof of this result uses Delaunay's triangulation to find sets of three points that can contained in the same disk and such that no other points belong to this disk.

The last part of this chapter focuses on the complexity when the radius of the disk is fixed. We show that the problem of determining $\gamma_D^{\mathrm{ID}}(\mathcal{P})$ is NP-complete when the radius is fixed, using a reduction from P_3-Partition-Grid. We then study the complexity when the points are colinear and show that the size of the minimal identifying set of disk of fixed radius can be determined in linear time :

Theorem 5.3 (G. et Parreau, 2019 [9]). Min Identification-Disk*(r) can be solved in linear time if* $\mathcal{P}$ *is a set of colinear points.*

To show this theorem we first prove that there always exists a minimal identifying set of disks in a particular configuration that we define as *normal form* and then give a greedy algorithm to find a smallest identifying set of disks in normal form.

This result is particularly interesting since this problem is close to the problem of identifying codes in unit interval graphs for which the complexity is still unknown.

Conclusion

In the PhD thesis, we explored several problems related to vertex covering.

We first studied power domination on grids (chapter 2). We found the power domination number of triangular grids with triangular shape and an upper bound for square grids of dimension 3.

We then explored the Maker-Breaker domination game (chapter 3). The aim was to find which player has a winning strategy and how many moves were required to dominate the graph when Dominator wins. We proved that these problems are PSPACE-complete for bipartite and chordal graphs but of polynomial complexity for trees. Furthermore, finding the outcome of the Maker-Breaker game is of polynomial complexity on cographs. In order to study this game, we introduced the notion of pairing dominating set and we proved that determining if a graph has a pairing dominating set was NP-complete. Two variants of the Maker-Breaker domination game were then introduced : the Maker-Breaker total domination game and the Avoider-Enforcer domination game. Finding the outcome of the total domination version is of polynomial complexity on cacti and finding the outcome of the Avoider-Enforcer version is of polynomial complexity on trees

Afterwards, we studied the strong geodetic number of graphs (chapter 4). We gave the exact value of the strong geodetic for complete bipartite graphs. Then, we introduce the strong geodetic core, which corresponds to the "efficiency" of a strong geodetic set. We studied the strong geodetic number for the Cartesian product of graphs and used the strong geodetic core to improve a previous bound. We then gave bounds for a particular case of Cartesian products - hypercubes.

Finally, we oriented our research to the identification of points in the plane using disks (chapter 5). After giving exact values and bounds in particular configurations, we studied bounds in the general case. In particular, we gave a bound of order $\frac{n}{3}$ when there are neither three colinear points nor four cocyclic ones. At last, we studied the complexity of the problem when the radius of the disks is fixed : we showed that the problem is NP-complete when there are no constraints on the disposition of points but that it is linear when the points are colinear.

Some of the problems studied in this manuscript lead to questions that would be interesting to investigate in the continuity of this PhD. Among the most interesting questions we have :

- Finding the value (at least asymptotically) of the power domination number of square grids of dimension 3 and above.
- Determining the complexity of the Maker-Breaker domination number on interval graphs.
- Studying the link between pairing dominating sets and the Maker-Breaker domination number. In particular, it would be interesting to study classes of graphs which have outcome $\mathcal{D}$ if and only if the game admits a pairing dominating set (as it is the case for trees and cographs).
- Continuing the study of the Avoider-Enforcer domination game and obverse the links with the Maker-Breaker version. Indeed, for graphs that we studied in both cases, Enforcer wins when Dominator wins and, reciprocally, Avoider wins when Staller wins. It would be interesting to caracterize the graphs for which the equivalence between the two games holds.
- Studying the complexity of the problem Strong Geodetic Set in which we want to know if a given set of vertices forms a strong geodetic set of a graph.
- Finding a lower bound on the strong geodetic number of Cartesian products of graphs.
- Finding the complexity of the problem of finding $\gamma_D^{ID}(P)$ in the general case.
- Studying sets of disks identifying points disposed uniformly at random in the square of side of length 1.
- Studying the identification using convex shapes different than disks and trying to adapt the method that we used to obtain Theorem 5.2 to these shapes.

Concerning long term perspectives, in addition to the study of the subjects presented in this PhD thesis, I would like to study other positional games on graphs with a combinatorial approach. In particular, I think that problems of covering of vertices are well suited for this type of games.

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