Power domination on triangular grids

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Abstract

The concept of *power domination* emerged from the problem of monitoring electrical systems. Given a graph G and a set $S \subseteq V(G)$, a set M of monitored vertices is built as follows: at first, M contains only the vertices of S and their direct neighbors, and then each time a vertex in M has exactly one neighbor not in M, this neighbor is added to M. The *power domination number* of a graph G is the minimum size of a set S such that this process ends up with the set M containing every vertex of G. We show that the power domination number of a triangular grid T_k with triangular-shaped border of length $k-1$ is $\lceil k/4 \rceil$, and the one of a triangular grid H_k with hexagonal-shaped border of length $k-1$ is $\lceil k/3 \rceil$.

Power domination is a problem that arose from the context of monitoring electrical systems (see [8, 1]), and was reformulated in graph terms by Haynes et al. [7].

Given a graph G and a set $S \subseteq V(G)$, we build a set M as follows: at first, M is the closed neighborhood of S, *i.e.*, $M = N[S]$, and then iteratively a vertex u is added to M if u has a neighbor v in M such that $N[v] \setminus \{u\} \subseteq M$ (we say that v *propagates* to u). At the end of the process, we call M the set of vertices *monitored* by S. We say that G is *monitored* when all its vertices are monitored. The set S is a *power dominating set* of G if at the end of the process $M = V(G)$, and the minimum cardinality of such a set is the *power domination number* of G, denoted by $\gamma_P(G)$.

Power domination has been particularly well-studied on regular grids and their generalizations: the exact value of γ_P has been determined for the square grid [4] and other products of paths [3], for the hexagonal grid [5], as well as for cylinders and tori [2]. These results are particularly interesting in comparison with the ones on the same classes for (classical) domination: for example, the problem of finding the domination number of grid graphs $P_n \times P_m$ was a difficult problem which has only been solved recently [6]. They also rely heavily on propagation: it is generally sufficient to monitor (with adjacency alone) a small portion of the graph in order to propagate to the whole graph.

We continue the study of power domination in grid-like graphs by focusing on triangular grids with triangular-shaped border. Figure 1 gives an example of such a grid. A *triangular grid with triangular-shaped border* T_k has vertex set $V(T_k) = \{(x, y, z) \mid x, y, z \in [0..k-1], x+y+z = k-1\}$. Two vertices v and v' are adjacent if and only if $|v'_x - v_x| + |v'_y - v_y| + |v'_z - v_z| = 2$ (two vertices are adjacent if and only if exactly two of their coordinates differ by 1).

We prove the following result:

Theorem 1. *Let T_k be a triangular grid with an triangular-shaped border of length $k-1$. For all positive integers k,* $\gamma_P(T_k) = \left\lceil \dfrac{k}{4} \right\rceil$.

The proof of this theorem has two parts. First, we show that the value of the γ_P is at most the one of the theorem by exhibiting a set S that reaches this value. Figure 1 shows an example of how to build such a set for T_8. Then, we give a sketch of the proof for the lower bound.

We also prove the following similar result for triangular grids with hexagonal-shaped border, for which the proof will not be detailed.

Theorem 2. *Let H_k be a triangular grid with a regular hexagonal-shaped border of length $k-1$. For all positive integers k, $\gamma_P(H_k) = \left\lceil \frac{k}{3} \right\rceil$.*

1 Upper bound

Lemma 1. *For every positive integer $k > 4$, $\gamma_P(T_k) \leq \left\lceil \frac{k}{4} \right\rceil$.*

Proof. First, note that it is sufficient to monitor the line $x = 0$ of the grid (*i.e.*, the bottom-line) in order to monitor the whole grid. Indeed, if it is the case, the vertex with coordinates $(0, k-1, 0)$ can propagate to the vertex with coordinates $(1, k-2, 0)$, then the one with coordinates $(0, k-2, 1)$ can propagate and so on until the line $x = 1$ is also monitored. Then, by using the same algorithm, the line $x = 1$ propagates to the line $x = 2$, and so on until the whole grid is monitored.

Let $\alpha = \lfloor k/4 \rfloor$. Let S' be the set defined as follows: $S' = \{v = (x, y, z) \mid x = 1, y = 1 + 4i, 0 \leq i < \alpha\}$. If $k \equiv 0 \mod 4$, then $S = S'$. Otherwise, let $S = S' \cup \{(1, k-2, 0)\}$ (one can easily check that in that case, the vertex $(1, k-2, 0)$ is not already in the set S'). We have $|S| = \lceil k/4 \rceil$. Then, $M = N[S]$ and one round of propagation is sufficient to monitor the first line of the triangle, which implies that the whole graph will be monitored (see Figure 1). □

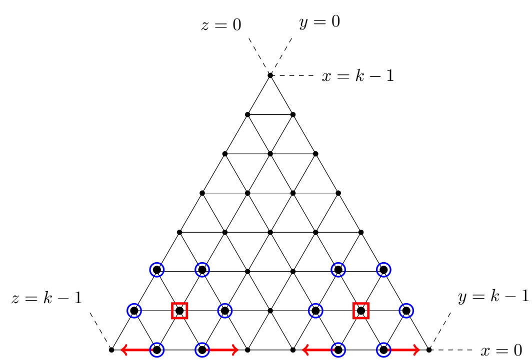

Figure 1: A minimal power dominating set for the grid T_8. Square vertices are in S and circled vertices are in $N[S]$. After one round of propagation, symbolized by the arrows, the line $x = 0$ is fully monitored.

2 Lower bound

To prove the lower bound we will study a quantity related to the edges on the border of the set of monitored vertices. We will see how this quantity evolves during the propagation. To define this quantity, we introduce the notions of *tip edges*, *base edges* and *holes* of a set of vertices.

By definition of T_k, two adjacent vertices u and v share a coordinate c_i and, if they are not both on the border, have two common neighbors: one for which c_i increases relative to u and v and one for which it decreases. We call the neighbor for which c_i decreases their *base neighbor* and the other one their *tip neighbor*.

Definition 1. *Given a set M of vertices:*

- *An edge uv is a* base edge *if u and v are in M and their tip neighbor is not in M (in particular, if u and v are on a border, uv is a base edge).*
- *An edge uv is a* tip edge *if u and v are in M and their base neighbor is not in M.*
- *A* hole *is a connected component of $V \setminus M$ that does not contain points of the border of the grid.*

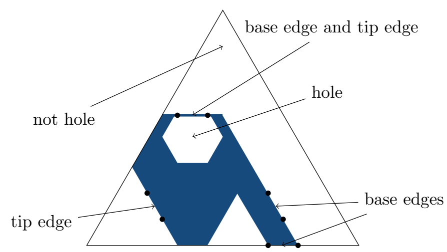

Figure 2: Examples of tip edges, base edges and holes. The set M is in blue.

We denote by $T(M)$ the set of tip edges, $B(M)$ the set of base edges, $h(M)$ the number of holes of M and $c(M)$ the number of connected components of M. We define the quantity Q as follows:

$$Q(M) = 2|T(M)| + |B(M)| + 3c(M) - 3h(M)\,.$$

The proof of the lower bound is now decomposed into three steps :

1. Prove that $Q(M)$ can only decrease as M increases with the propagation
2. Compute $Q(M)$ before the propagation begins
3. Compute $Q(M)$ after the whole graph is monitored

Step 3 comes directly from the definition. We have $Q(V) = 3k$ indeed there are $3(k-1)$ edges on the border and only one connected component.

Steps 1 and 2 come from the following Lemmas 2 and 3.

Lemma 2. *Let $M[i]$ be the set of monitored vertices after i propagation steps. Then $Q(M[i+1]) \leq Q(M[i])$.*

This Lemma is proved by a case study of the different configurations that can lead to a propagation. Figure 3 shows some of these cases. In these cases, we can see that each time some edges are added to $T(M)$ or $B(M)$, this is compensated by the fact that some edges are removed, holes are created or several connected component are connected and become only one.

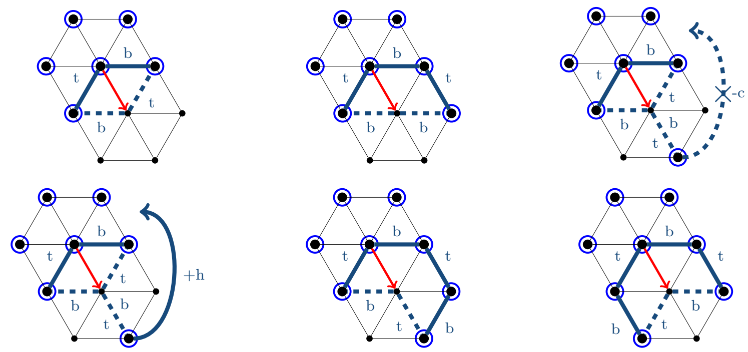

Figure 3: Some cases of the proof of Lemma 2. Propagation is shown by the arrow. Circled vertices are in M. The edges removed from $T(M)$ and $B(M)$ are drawn thicker. The edges added to $T(M')$ and $B(M')$ are dashed. The edges marked with b or t are base edges and tip edges, respectively. The curved arrows labeled by $+h$ or $-c$ represent the creation of a hole or the junction of two different connected components.

Lemma 3. *Let S be a set of vertices of a triangular grid T_k. Then $Q(N[S]) \leq 12|S|$.*

To prove this lemma, we can assume first that $N[S]$ is connected, since otherwise we can prove the result independently for each connected component. Next we build an auxiliary graph G_S based on the structure of S in the triangular grid. We prove that G_S is planar and apply Euler's formula to it. Then, we use a discharging method to prove the inequality $2|T(N[S])|+|B(N[S])| \leq 9|S|+3|E_S|$. The result follows.

By applying these lemmas we have that $12|S| \geq Q(N[S]) \geq Q(V) = 3k$ and we can deduce the following lower bound on $\gamma_P(T_k)$:

Lemma 4. *For all positive integers $k > 4$, $\gamma_P(T_k) \geq \left\lceil \frac{k}{4} \right\rceil$.*

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